

The Gough-James Theory of Quantum Feedback Networks in the Belavkin Representation

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Abstract

The mathematical theory of quantum feedback networks has recently been developed by Gough and James [5] for general open quantum dynamical systems interacting with bosonic input fields. In this article we show, that their feedback reduction formula for the coefficients of the closed-loop quantum stochastic differential equation can be formulated in terms of Belavkin matrices. We show that the reduction formula leads to a non-commutative Möbius transformation based on Belavkin matrices, and establish a \star -unitary version of the Siegel identities.

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1 Introduction

In a recent publication [5] Gough and James have introduced a model for a quantum feedback network. Each component may be modelled in isolation as a Hudson-Parthasarathy model, valid for quantum optical models, and represented as a single vertex with an equal number of external inputs and outputs carried along semi-infinite transmission lines represented as directed edges. The algebraic procedure is to collect all the operator coefficients of the associated quantum stochastic differential equation governing all components into a matrix. This gives the open-loop description, and feedback is introduced by connecting various input/output lines to give internal edges. They use a Hamiltonian model for the entire network which generalizes the Chebotarev-Gregoratti Hamiltonian which describes the propagation of the fields along the edges and their interaction at the vertices. They obtain a Markovian limit for the network in a zero time delay limit, eliminating the internal edges in the process. The limit quantum stochastic differential equation is then described by a reduced operator matrix.

We show that their feedback reduction formula is a Möbius transformation that can naturally be extended to mappings into the Belavkin matrix representation of quantum stochastic calculus. In particular we interpret this as a non-commutative Möbius transformation between \star -unitaries. We begin by recalling the basic notions of quantum stochastic calculus and its Belavkin formulation.

1.1 Quantum Stochastic Processes

The Hudson-Parthasarathy theory of quantum stochastic calculus considers quantum stochastic processes as operator valued processes on Hilbert spaces of the form $\mathfrak{H} = \mathfrak{h}_0 \otimes \Gamma(\mathfrak{k} \otimes L^2[0, \infty))$ where \mathfrak{h}_0 is a fixed Hilbert space, called the *initial space*, and \mathfrak{k} is a fixed Hilbert space called the *internal space*. We shall be interested in the finite dimensional case $\mathfrak{k} = \mathbb{C}^n$ where $n \geq 1$. Here $\Gamma(t)$ denotes the second quantization functor to (Bosonic) Fock space. We shall denote the time variable t by A_t^{00} . Taking $\{|e_i\rangle : i = 1, \dots, n\}$ to be an orthonormal basis for \mathfrak{k} , the creation process to state $|e_i\rangle$ will be denoted as A_t^{i0} , while its adjoint, the annihilator for the state, is denoted as A_t^{0i} . The scattering process from state $|e_j\rangle$ to state $|e_i\rangle$ will be denoted as A_t^{ij} . In this way, we have the $(1+n) \times (1+n)$ *fundamental quantum processes* $A_t^{\alpha\beta}$. (We adopt the convention that Latin indices range over $1, \dots, n$ while Greek indices range over $0, 1, \dots, n$. We also apply a summation convention for repeated indices over the corresponding ranges.) We note that we have $(A_t^{\alpha\beta})^\dagger = A_t^{\beta\alpha}$.

As is well known \mathfrak{H} decomposes as $\mathfrak{H}_{[0,t]} \otimes \mathfrak{H}_{(t,\infty)}$ for each $t > 0$ where $\mathfrak{H}_{[0,t]} = \mathfrak{h}_0 \otimes \Gamma(\mathfrak{k} \otimes L^2[0, t])$ and $\mathfrak{H}_{(t,\infty)} = \Gamma(\mathfrak{k} \otimes L^2(t, \infty))$. We shall write \mathfrak{A}_t for the space of operators on \mathfrak{H} that act trivially on the future component $\mathfrak{H}_{(t,\infty)}$. A quantum stochastic process $X_t = \{X_t : t \geq 0\}$ is said to be *adapted* if $X_t \in \mathfrak{A}_t$ for each $t \geq 0$.

Taking $\{x_{\alpha\beta}(t) : t \geq 0\}$ to be a family of adapted quantum stochastic processes, we may then form their quantum stochastic integral $X_t = \int_0^t x_{\alpha\beta}(s) dA_s^{\alpha\beta}$ where the differentials are understood in the Itô sense. Given a similar quantum Itô integral Y_t , with $dY_t = y_{\alpha\beta}(t) dA_t^{\alpha\beta}$, we have the quantum Itô product rule

$$d(X_t \cdot Y_t) = dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t, \quad (1)$$

with the Itô correction given by

$$dX_t \cdot dY_t = x_{\alpha k}(t) y_{k\beta}(t) dA_t^{\alpha\beta}. \quad (2)$$

The coefficients $\{x_{\alpha\beta}(t)\}$ may be assembled into a matrix

$$\mathbf{X}_t = \left(\begin{array}{c|c} x_{00}(t) & x_{0\bullet}(t) \\ \hline x_{\bullet 0}(t) & x_{\bullet\bullet}(t) \end{array} \right) \in \mathfrak{A}_t^{(1+n) \times (1+n)}, \quad (3)$$

which we call the *Itô matrix* for the process. (Here we use the convention that $x_{0\bullet}(t)$ denotes the row vector with entries $(x_{0j}(t))_{j=1}^n$, etc. The Itô matrix for a product $X_t Y_t$ of quantum Itô integrals will then have entries $\{x_{\alpha\beta}(t) Y_t + X_t y_{\alpha\beta}(t) + x_{\alpha k}(t) y_{k\beta}(t)\}$ and is therefore given by $\mathbf{X}_t Y_t + X_t \mathbf{Y}_t + \mathbf{X}_t \mathbf{P} \mathbf{Y}_t$, where $\mathbf{P} := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{I}_n \end{array} \right)$.

1.2 Belavkin's Matrix Representation

We consider the mapping from Itô matrices $\mathbf{X} \in \mathfrak{A}^{(1+n) \times (1+n)}$ to associated *Belavkin matrices*

$$\mathbb{X} = \left(\begin{array}{c|c|c} 0 & x_{0\bullet} & x_{00} \\ \hline 0 & x_{\bullet\bullet} & x_{\bullet 0} \\ \hline 0 & 0 & 0 \end{array} \right) \in \mathfrak{A}^{(1+n+1) \times (1+n+1)}. \quad (4)$$

We also introduce

$$\mathbb{I}_n := \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \mathbb{I}_n & 0 \\ \hline 0 & 0 & 1 \end{array} \right), \quad \mathbb{J}_n := \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & \mathbb{I}_n & 0 \\ \hline 1 & 0 & 0 \end{array} \right),$$

where \mathbb{I}_n is the $n \times n$ identity matrix. The subscripts n will be generally dropped from now on for convenience. We have the following identifications

$$\begin{aligned} \mathbf{X}^\dagger &\longleftrightarrow \mathbb{X}^\star := \mathbb{J}\mathbb{X}^\dagger\mathbb{J}, \\ \mathbf{X}\mathbf{P}\mathbf{Y} &\longleftrightarrow \mathbb{X}\mathbb{Y}, \\ \mathbf{X}\mathbf{Y} &\longleftrightarrow \mathbb{X}\mathbb{J}\mathbb{Y}. \end{aligned}$$

We shall refer to \mathbb{X}^\star as the \star -*involution* of \mathbb{X} . The Itô differential $dX_t = x_{\alpha\beta}(t) dA_t^{\alpha\beta}$ may then be written as

$$dX_t = \text{tr} \left\{ \mathbb{X}_t d\tilde{\mathbb{A}}_t \right\},$$

where (with \prime denoting the usual transpose for arrays)

$$d\tilde{\mathbb{A}}_t := \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline (dA^{0\bullet})' & (dA^{\bullet\bullet})' & 0 \\ \hline dA^{00} & (dA^{\bullet 0})' & 0 \end{array} \right).$$

The main advantage of using this representation is that the Itô correction $\mathbf{X}\mathbf{P}\mathbf{Y}$ can now be given as just the ordinary product $\mathbb{X}\mathbb{Y}$ of the Belavkin matrices.

Let X_t and Y_t be quantum stochastic integrals, then the quantum Itô product rule may be written as

$$d(X_t Y_t) = \text{tr} \left\{ [(X_t \mathbb{I} + \mathbb{X}_t)(Y_t \mathbb{I} + \mathbb{Y}_t) - (X_t Y_t) \mathbb{I}] d\tilde{\mathbb{A}}_t \right\}. \quad (5)$$

The process $f(X_t)$ has differential $df(X_t) = \text{tr} \{ [f(X_t \mathbb{I} + \mathbb{X}_t) - f(X_t) \mathbb{I}] d\tilde{\mathbb{A}}_t \}$.

1.3 Evolutions and Dynamical Flows

Hudson and Parthasarathy [11] show that the quantum stochastic differential equation (QSDE)

$$dV_t = \text{tr} \left\{ \mathbb{G} V_t d\tilde{\mathbb{A}}_t \right\}, \quad V_0 = 1, \quad (6)$$

has a unique solution for a given constant Belavkin matrix $\mathbb{G} = \mathbb{V} - \mathbb{I}$ of coefficients on $\mathfrak{B}(\mathfrak{h}_0)$, the bounded operators on \mathfrak{h}_0 . Necessary and sufficient conditions for unitarity are then given in Belavkin representation by

$$(\mathbb{I} + \mathbb{G})(\mathbb{I} + \mathbb{G})^* = \mathbb{I} = (\mathbb{I} + \mathbb{G})^*(\mathbb{I} + \mathbb{G}).$$

This states that $\mathbb{V} = \mathbb{I} + \mathbb{G}$ is \star -unitary on $\mathfrak{B}(\mathfrak{h}_0)^{(1+n+1) \times (1+n+1)}$, that is

$$\mathbb{V}\mathbb{V}^* = \mathbb{I} = \mathbb{V}^*\mathbb{V}. \quad (7)$$

We may write the QSDE as $dV_t = \text{tr} \left\{ [\mathbb{V}V_t - \mathbb{I}V_t] d\tilde{\mathbb{A}}_t \right\}$.

Lemma 1 *The most general form for \mathbb{V} leading to a unitary is*

$$\mathbb{V} = \left(\begin{array}{c|c|c} 1 & -L^\dagger S & -\frac{1}{2}L^\dagger L - iH \\ \hline 0 & S & L \\ \hline 0 & 0 & 1 \end{array} \right) \equiv \left(\begin{array}{c|c|c} V_{00} & V_{0k} & V_{00'} \\ \hline V_{k0} & V_{kk} & V_{k0'} \\ \hline V_{0'0} & V_{0'k} & V_{0'0'} \end{array} \right). \quad (8)$$

where S is a unitary in $\mathfrak{B}(\mathfrak{h}_0)^{n \times n}$, L is a column vector length n with entries in $\mathfrak{B}(\mathfrak{h}_0)$, and H is self-adjoint in $\mathfrak{B}(\mathfrak{h}_0)$.

The proof follows from the analysis of [11]. The triple (S, L, H) is termed the Hudson-Parthasarathy parameters of the open system evolution. In standard notation the QSDE reads as (summ over all field multiplicities)

$$dV_t = \left\{ (S_{jk} - \delta_{jk}) dA_t^{jk} + L_j dA_t^{j0} - L_j^* S_{jk} dA_t^{0j} - \left(\frac{1}{2} L_j^* L_j + iH \right) dA_t^{00} \right\} V_t.$$

We interpret $A_t^{\alpha\beta}$ as the input noise and $V_t^* A_t^{\alpha\beta} V_t$ as the output noise.

2 Quantum Cascaded Systems

If two systems are cascaded in series then the Hudson-Parthasarathy parameters of the composite system were shown to be [6],[5]

$$\begin{aligned} S_{\text{series}} &= S_2 S_1, \\ L_{\text{series}} &= L_2 + S_2 L_1, \\ H_{\text{series}} &= H_1 + H_2 + \text{Im} \left\{ L_2^\dagger S_2 L_1 \right\}. \end{aligned}$$

Here the output of the first system (S_1, L_1, H_1) is fed forward as the input to the second system (S_2, L_2, H_2) and the limit of zero time delay is assumed. As remarked in [5], the series product actually arises natural in Belavkin matrix form as

$$\mathbb{V}_{\text{series}} = \mathbb{V}_2 \mathbb{V}_1.$$

The product is clearly associative, as one would expect physically, and the general rule for several systems in series is then $\mathbb{V}_{\text{series}} = \mathbb{V}_n \cdots \mathbb{V}_2 \mathbb{V}_1$

3 General Feedback Reduction Formula

The internal edges may be eliminated in a zero time delay limit to obtain a reduced model. Let $0 < n_i < n$ be then number of internal edges to be eliminated, and let $n_e = n - n_i$ be the remaining edges. The algebraic information about the original network is contained in the matrix \mathbb{V} which we partition as

$$\mathbb{V} = \left(\begin{array}{c|cc|c} V_{00} & V_{0e} & V_{0i} & V_{00'} \\ \hline V_{e0} & V_{ee} & V_{ei} & V_{e0'} \\ V_{i0} & V_{ie} & V_{ii} & V_{i0'} \\ \hline V_{0'0} & V_{0'e} & V_{0'i} & V_{0'0'} \end{array} \right).$$

Here we decompose indices into two groups **e** and **i** distinguishing external and internal. That is, $V_{00} = V_{0'0'} = 1$, $V_{e0} = V_{i0} = V_{0'0} = V_{0'e} = V_{0'i} = 0$

$$\begin{aligned} S &= \begin{bmatrix} V_{ee} & V_{ei} \\ V_{ie} & V_{ii} \end{bmatrix} = \begin{bmatrix} S_{ee} & S_{ei} \\ S_{ie} & S_{ii} \end{bmatrix}, \\ L &= \begin{bmatrix} V_{e0'} \\ V_{i0'} \end{bmatrix} = \begin{bmatrix} L_e \\ L_i \end{bmatrix}, \end{aligned}$$

and $[V_{0e} \ V_{0i}] = -SL^*$.

Theorem 2 *We assemble a Belavkin matrix $\mathcal{F}(\mathbb{V}, X)$ in $\mathfrak{A}^{(1+n_e+1) \times (1+n_e+1)}$ with sub-blocks*

$$\mathcal{F}(\mathbb{V}, X)_{\alpha\beta} = V_{\alpha\beta} + V_{\alpha i} X (1 - V_{ii} X)^{-1} V_{i\beta}$$

for $\alpha = 0, e, 0'$ and $\beta = 0, e, 0'$, where we fix a unitary operator in $X \in \mathbb{C}^{n_i \times n_i}$ such that the inverse above exists. Then $\mathcal{F}(\mathbb{V}, X)$ is again a \star -unitary, that is,

$$\mathcal{F}(\mathbb{V}, X)^* \mathcal{F}(\mathbb{V}, X) = \mathcal{F}(\mathbb{V}, X) \mathcal{F}(\mathbb{V}, X)^* = \mathbb{I}, \quad (9)$$

so that $\mathcal{F}(\mathbb{V}, X)$ determines a unitary dynamics for the reduced set of n_e inputs. Moreover, we have the identity

$$\mathcal{F}(\mathbb{V}, X)^* = \mathcal{F}(\mathbb{V}^*, X^\dagger). \quad (10)$$

Remark 3 *The matrix X appearing above is typically an adjacency matrix in applications, describing which internal outputs are to be connected to which internal inputs. In engineering, it could be interpreted as a gain matrix. We also point out that the involutions in (10) are on spaces of different dimensions. The first involves \mathbb{J}_{n_i} while the second involves \mathbb{J}_n .*

Proof. The construction of $\mathcal{F}(\mathbb{V}, X)$ is essentially the rephrasing of the Möbius transformation associated with the reduction, introduced in [5], in the language of Belavkin matrices. The construction in (8) clearly yields a Belavkin

matrix over the remaining n_e external degrees of freedom. It is a straightforward calculation to show that, for X unitary, the matrix takes the form

$$\mathcal{F}(\mathbb{V}, X) = \left(\begin{array}{c|c|c} 1 & -L^{\text{red}\dagger} S & -\frac{1}{2} L^{\text{red}\dagger} L^{\text{red}} - i H^{\text{red}} \\ \hline 0 & S^{\text{red}} & L^{\text{red}} \\ \hline 0 & 0 & 1 \end{array} \right),$$

with the Hudson-Parthasarathy parametrizing operators $(S^{\text{red}}, L^{\text{red}}, H^{\text{red}})$ given by

$$\begin{aligned} S^{\text{red}} &= S_{ee} + S_{ei} (X^{-1} - S_{ii})^{-1} S_{ie}, \\ L^{\text{red}} &= L_e + S_{ei}^{-1} (X^{-1} - S_{ii}) L_i, \\ H^{\text{red}} &= H + \sum_{i=e, e'} \text{Im} L_j^\dagger S_{ji}^{-1} (X^{-1} - S_{ii}) L_i, \end{aligned}$$

in agreement with [5]. ■

For $\mathbb{X} \in \mathfrak{A}^{(1+n+1) \times (1+m+1)}$ we shall introduce the extended convention $\mathbb{X}^\star := \mathbb{J}_m \mathbb{X}^\dagger \mathbb{J}_n$. Let \mathbb{V} be the Belavkin matrix generating a unitary quantum dynamics as above, we define the Möbius transformation $\Phi : D \mapsto \mathfrak{A}^{(1+n_e+1) \times (1+n_e+1)}$ by $\Phi = \mathcal{F}(\mathbb{V}, \cdot)$ with domain $D = \{X \in \mathbb{C}^{n_i \times n_i} : \mathbb{I} - V_{ii} X \text{ is invertible}\}$.

Theorem 4 *The mapping Φ satisfies the Siegel type identities*

$$\begin{aligned} \Phi(X)^\star \Phi(Y) &= \mathbb{I} + \begin{pmatrix} V_{0i} \\ V_{ei} \\ V_{0'i} \end{pmatrix}^\star \left(1 - X^\dagger V_{ii}^\dagger\right)^{-1} (X^\dagger Y - 1) (1 - V_{ii} Y)^{-1} \begin{pmatrix} V_{0i} \\ V_{ei} \\ V_{0'i} \end{pmatrix}, \\ \Phi(X) \Phi(Y)^\star &= \mathbb{I} + (V_{i0}, V_{ie}, V_{i0'}) (1 - X V_{ii})^{-1} (X Y^\dagger - 1) (1 - V_{ii}^\dagger Y)^{-1} (V_{i0}, V_{ie}, V_{i0'})^\star. \end{aligned}$$

In particular, Φ maps unitaries to \star -unitaries.

Proof. The form of these relations are similar to the standard Siegel identities, see for instance [14], but with the \star -involution now replacing the usual \dagger . The algebraic manipulations involved are otherwise identical. ■

We remark that the standard Siegel type identities have independently been extended in an entirely different direction to deal with Bogoliubov transformations in a recent paper of Gough, James and Nurdin [15]. They replace the usual \dagger -involution with an alternative involution, this time on the space of doubled up matrices required to describe the symplectic structure, however they similarly rely on the argument used in the above proof.

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